SOLVING THE GRAVITY FLOW EQUATION FOR FLOW RATE USING THE NEWTON-RAPHSON ITERATION TECHNIQUE

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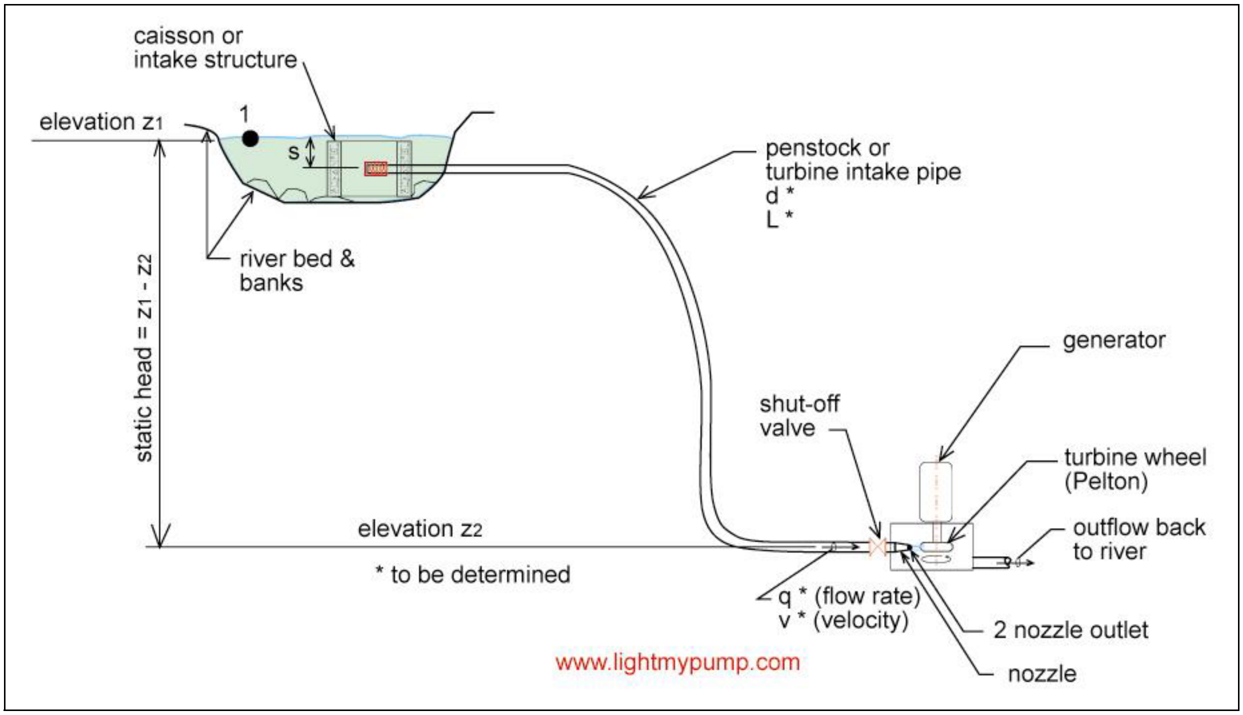

Figure 1. Typical micro-hydro installation.

Imperial Units

The equation for flow vs. static head and friction for a system that provides a water jet used as the motive force for an impulse turbine is:

$$\frac{0.4085 \times q(gpm)}{N \times d_N^2 (in)^2} = \left(2 \times 32.17 \times (H(ft) - H_F(ft)(q)\right)^{1/2} \quad \textbf{(1)}$$

where q is the flow rate in gals./min, N the number of nozzle (1 to 4), d_N the nozzles (0.25 to 1) diameter in inch, H the static head in feet, and H_F the total friction loss in feet.

The friction loss $H_F(q)$ is based on the Darcy/Weisback equation:

$$H_F(ft)(q) = \frac{1200 \times v_p^2 (ft/s)^2}{2 \times 32.17 \times d_p(in)} \times \frac{L(ft)}{100} \times f$$

where v_p is the velocity in the pipe in ft/s, d_p the diameter of the pipe in inch, L the length of the pipe in feet and f the friction factor or parameter (non-dimensional).

The friction parameter is given by the Swamee-Jain equation:

$$f = \frac{0.25}{\left(\log\left(\frac{\varepsilon(in)}{3.7 \times d_p(in)} + \frac{5.74}{\mathrm{Re}^{0.9}}\right)\right)^2}$$

where ε is the RMS roughness of the surface in inches, and Re the Reynolds number(non-dimensional).

$$\mathrm{Re} = \frac{7745.8 \times v_p(ft/s) \times d_p(in)}{\nu(cSt)}$$

where ν is the kinematic viscosity of the fluid, for water it is 1 centiStoke.

Velocity can be expressed as a function of flow:

$$v_p = \frac{0.4085 \times q(gpm)}{d_p^2(in)^2}$$

therefore *Re* is:

$$\mathrm{Re} = \frac{7745.8 \times 0.4085 \times q(gpm)}{\nu(cSt) \times d_p(in)}$$

$$f(q) = \frac{0.25}{(\log(\frac{\varepsilon(in)}{3.7 \times d_p(in)} + \frac{5.74}{\left(\frac{7745.8 \times 0.4085 \times q(gpm)}{\nu(cSt) \times d_p(in)}\right)^{0.9}}))^2}$$

and the friction loss $H_F(q)$ can then be expressed as:

$$H_F(ft)(q) = \frac{1200 \times 0.4085^2 \times 0.25}{2 \times 32.17 \times d_p^5 (in)^5} \times \frac{L(ft)}{100} \times \frac{q^2 (gpm)^2}{(\log(\frac{\varepsilon(in)}{3.7 \times d_p(in)} + \frac{5.74}{\left(\frac{7745.8 \times 0.4085 \times q(gpm)}{\nu(cSt) \times d_p(in)}\right)^{0.9}}))^2}$$

and if we define the following expression as K_1:

$$K_1 = \frac{1200 \times 0.4085^2}{2 \times 32.17 \times d_p^5 (in)^5} \times \frac{L(ft)}{100}$$

then $H_F(q)$ becomes:

$$H_F(ft)(q) = K_1 \times \frac{q^2 (gpm)^2}{(\log(\frac{\varepsilon(in)}{3.7 \times d_p(in)} + \frac{5.74}{\left(\frac{7745.8 \times 0.4085 \times q(gpm)}{\nu(cSt) \times d_p(in)}\right)^{0.9}}))^2}$$

or

$$H_F(ft)(q) = K_1 \times f \times q^2 (gpm)^2$$

Normally to solve equation (1) we would create a function I call G and apply the the Newton-Raphson iteration technique :

$$G = \frac{0.4085 \times q(gpm)}{N \times d_N^2 (in)^2} - \left(2 \times 32.17 \times (H(ft) - H_F(ft)(q)\right)^{1/2}$$

however after some trials it became apparent that I could not get a convergence for certain values of q and this is because the term H_F would sometimes get larger than H and this would cause the iteration process to fail. It was suggested by a gentleman called Torsten Hennig on the math forum alt.math.undergrad at http://mathforum.org/kb/forum.jspa?forumID=56 that if I square both sides of equation (1) I would eliminate the problem and this turned out to be the solution.

Equation (1) then becomes:

$$\frac{0.4085^2 \times q^2 (gpm)^2}{N^2 \times d_N^4 (in)^4} = 2 \times 32.17 \times (H(ft) - H_F(ft)(q)$$

and we create a function F:

$$F = \frac{0.4085^2 \times q^2 (gpm)^2}{N^2 \times d_N^4 (in)^4} - 2 \times 32.17 \times (H(ft) - H_F(ft)(q)$$

that we can solve using the Newton-Raphson iteration technique.

A value for q will be found that will converge if we modify the initial value with the result of the calculation of the residue RES. In the case of the N-R technique the residue is:

$$RES = \frac{F}{dF/dq}$$

and the value of q for successive iterations will be $q_n = q_{n-1} - RES$ until the residue is very small and close to zero (less than 1 x 10^{-6})

$$\frac{dF}{dq} = 2 \times \frac{0.4085^2}{N^2 \times d_N^4 (in)^4} \times q(gpm) - 2 \times 32.17 \times \frac{dH_F(ft)(q)}{dq}$$

Here we make use of the derivative rule:

$$\frac{d\,\frac{f(q)}{g(q)}}{dq} = \frac{g(q)\frac{df(q)}{dq} - f(q)\frac{dg(q)}{dq}}{g^2(q)}$$

and

$$\frac{d((\log_{10} f(q))^2}{dq} = \frac{2 \times \log_{10} f(q) \times \log_{10} e}{f(q)} \times \frac{df(q)}{dq}$$

$$\frac{dH_F(ft)}{dq} = K_1 \times \left(f^2 \times 2 \times q(gpm) - \left(\frac{2 \times 0.25^{-0.5} \times f^{1.5} \times \log_{10} e \times 5.74 \times -0.9 \times \left(\frac{7745.8 \times 0.4085}{\nu(cSt) \times d_p^2 (in)^2} \right)^{-0.9} \times q(gpm)^{0.1}}{10^{\left(\frac{0.25}{f}\right)^{0.5}}} \right) \right)$$

and if we define the following expression as K_2:

$$K_2 = 2 \times 0.25^{-0.5} \times \log_{10} e \times 5.74 \times -0.9 \times \left(\frac{7745.8 \times 0.4085}{\nu(cSt) \times d_p^2 (in)^2} \right)^{-0.9}$$

then

$$\frac{dH_F}{dq} = K_1 \times \left(f(q)^2 \times 2 \times q(gpm) - \left(\frac{K_2 \times f(q)^{1.5} \times q(gpm)^{0.1}}{10^{\left(\frac{0.25}{f(q)}\right)^{0.5}}} \right) \right)$$

and

F(q) is the equation to be solved and must equal zero for the appropriate value of q.

$$F(q) = \frac{0.4085 \times q(gpm)}{N \times d_N^2(in)^2} - \left(2 \times 32.17 \times (H(ft) - H_F(ft)(q)\right)^{1/2} \quad \textbf{(2)}$$

and H_F is given by:

$$H_F(q) = K_1 \times f(q) \times q(gpm)^2 \quad \textbf{(3)}$$

and f(q) by:

$$f(q) = \frac{0.25}{(\log(\frac{\varepsilon(in)}{3.7 \times d_p(in)} + \frac{5.74}{\left(\frac{7745.8 \times 0.4085 \times q(gpm)}{\nu(cSt) \times d_p(in)}\right)^{0.9}}))^2} \quad \textbf{(4)}$$

We want to solve equation (2) for F(q) = 0 based on the values of the terms in equations (3) and (4).

Using the N-R iteration technique we will need the values of df/dq and RES given below.

$$\frac{dF}{dq} = 2 \times \frac{0.4085^2}{N^2 \times d_N^4(in)^4} \times q(gpm) - 2 \times 32.17 \times \frac{dH_F(ft)(q)}{dq}$$

$$RES = \frac{F}{dF/dq}$$

Metric Units

The equation for flow vs. static head and friction for a system that provides a water jet used as the motive force for an impulse turbine is:

$$\frac{21.22\times q(L/\min)}{N\times d_N^2(mm)^2}=\left(2\times 9.81\times(H(m)-H_F(m)(q))\right)^{1/2} \quad \textbf{(1)}$$

where q is the flow rate in liters/min, N the number of nozzle (1 to 4), d_N the nozzles (0.25 to 1) diameter in millimeters, H the static head in meters, and H_F the total friction loss in meters.

The friction loss $H_F(q)$ is based on the Darcy/Weisback equation:

$$H_F(m)(q)=\frac{10^5\times v_p^2(m/s)^2}{2\times 9.81\times d_p(mm)}\times\frac{L(m)}{100}\times f$$

where v_p is the velocity in the pipe in m/s, d_p the diameter of the pipe in mm, L the length of the pipe in meter and f the friction factor or parameter (non-dimensional).

The friction parameter is given by the Swamee-Jain equation:

$$f=\frac{0.25}{\left(\log\left(\frac{\varepsilon(mm)}{3.7\times d_p(mm)}+\frac{5.74}{\mathrm{Re}^{0.9}}\right)\right)^2}$$

where ε is the RMS roughness of the surface in mm, and Re the Reynolds number(non-dimensional).

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$$f(q) = \frac{0.25}{(\log(\frac{\varepsilon(mm)}{3.7 \times d_p(mm)} + \frac{5.74}{\left(\frac{1000 \times 21.22 \times q(L/\min)}{\nu(cSt) \times d_p(mm)}\right)^{0.9}}))^2}$$

and the friction loss $H_F(q)$ can then be expressed as:

$$H_F(m)(q) = \frac{10^5 \times 21.22^2 \times 0.25}{2 \times 9.81 \times d_p^5(mm)^5} \times \frac{L(m)}{100} \times \frac{q^2(L/\min)^2}{(\log(\frac{\varepsilon(mm)}{3.7 \times d_p(mm)} + \frac{5.74}{\left(\frac{1000 \times 21.22 \times q(L/\min)}{\nu(cSt) \times d_p(mm)}\right)^{0.9}}))^2}$$

and if we define the following expression as K_1:

$$K_1 = \frac{10^5 \times 21.22^2}{2 \times 9.81 \times d_p^5(mm)^5} \times \frac{L(m)}{100}$$

then $H_F(q)$ becomes:

$$H_F(m)(q) = K_1 \times \frac{q^2(L/\min)^2}{(\log(\frac{\varepsilon(mm)}{3.7 \times d_p(mm)} + \frac{5.74}{\left(\frac{1000 \times 21.22 \times q(L/\min)}{\nu(cSt) \times d_p(mm)}\right)^{0.9}}))^2}$$

or

$$H_F(m)(q) = K_1 \times f \times q^2(L/\min)^2$$

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$$\frac{dF}{dq} = 2 \times \frac{21.22^2}{N^2 \times d_N^4 (mm)^4} \times q(L/\min) - 2 \times 9.81 \times \frac{dH_F(m)(q)}{dq}$$

Here we make use of the derivative rule:

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$$\frac{d((\log_{10} f(q))^2}{dq} = \frac{2 \times \log_{10} f(q) \times \log_{10} e}{f(q)} \times \frac{df(q)}{dq}$$

$$\frac{dH_F(m)}{dq}=K_1\times\left(f^2\ \times 2\times q(L/\min)-\left(\frac{2\times 0.25^{-0.5}\times f^{1.5}\times\log_{10}e\times 5.74\times -0.9\times\left(\frac{1000\times 21.22}{\nu(cSt)\times d_p^2(mm)^2}\right)^{-0.9}\times q(L/\min)^{0.1}}{10^{\left(\frac{0.25}{f}\right)^{0.5}}}\right)\right)$$

and if we define the following expression as K_2:

$$K_2=2\times 0.25^{-0.5}\times\log_{10}e\times 5.74\times -0.9\times\left(\frac{1000\times 21.22}{\nu(cSt)\times d_p^2(mm)^2}\right)^{-0.9}$$

then

$$\frac{dH_F}{dq}=K_1\times\left(f(q)^2\ \times 2\times q(L/\min)\ -\left(\frac{K_2\ \times f(q)^{1.5}\times q(L/\min)^{0.1}}{10^{\left(\frac{0.25}{f(q)}\right)^{0.5}}}\right)\right)$$

and

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