Comparison of the iterative approximations of the Colebrook-White equation

Here's a review of other formulas and a mathematically exact formulation that is valid over the entire range of *Re* values

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Friction factor estimation is a key component of piping system design and the Colebrook-White equation is typically the method of choice for computing turbulent flow friction factor in rough pipes:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) \tag{1}$$

It relates the friction factor, f, implicitly to the pipe roughness, ε/D, and the Reynolds number, Re. Because of the implicit nature of Eq. 1, graphical methods were originally proposed for f estimation[1] and are still used today. While the visual representation in a graphical correlation is certainly appealing, accurate f determination is difficult and this approach is not suited for most current computer-based piping system design projects.

For computer implementation, iterative numerical methods such as the Newton-Raphson method[2] can be used to determine f from Eq. 1. Ideally, these iterative calculations are not desirable, and in an attempt to simplify f estimation from Eq. 1, several explicit approximations of f have been proposed.[3–6] Accuracy of f values determined from these correlations varies greatly and not all correlations are valid over a large Re range (typically $4{,}000 < Re < 10^8$) to be universally applicable. Accuracy of the noniterative empirical correlations has been comprehensively evaluated[7] and was found to be in the 1.42–28.23% range compared with 1% error for a simplified form of a truly explicit representation of Eq. 1.

In addition to the noniterative correlations mentioned, several iterative approximations have also been proposed for Eq. 1.[4–6,8,9] These are more complex functional relationships between f, ε/D and Re but result in f values with higher accuracy. To completely eliminate need for empirical correlations, we have proposed an explicit, mathematically exact formulation of Eq. 1 that is valid over the entire range of Re values and results in highly accurate f values.[10,11] Accuracy of a simplified form of this formulation was presented earlier[7] and in this study we present a comparison of two other forms of this formulation with the various iterative approximations of Eq. 1.

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Details on the derivation of the explicit reformulation have been presented elsewhere[11] and only the final equations are shown here. The friction factor, f, can be explicitly related to ε/D and Re as:

$$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta\right] \tag{2}$$

where $a = \frac{2}{\ln(10)}$; $b = \frac{\varepsilon/D}{3.7}$; $d = \left(\frac{\ln(10)}{5.02}\right)Re$;

$q = s^{\left(s/(s+1)\right)}$; and $s = bd + \ln(d)$

Two different formulations are available for δ, the linear formulation, δ_{LA}, and the continued fractions formulation, δ_{CFA}, and they vary in complexity and accuracy:

$$\delta_{LA} = \left(\frac{g}{g+1}\right)z$$

$$\delta_{CFA} = \delta_{LA}\left(1 + \frac{z/2}{(g+1)^2 + (z/3)(2g-1)}\right) \tag{3}$$

where $g = bd + \ln\left(\frac{d}{q}\right)$ and $z = \ln\left(\frac{q}{g}\right)$

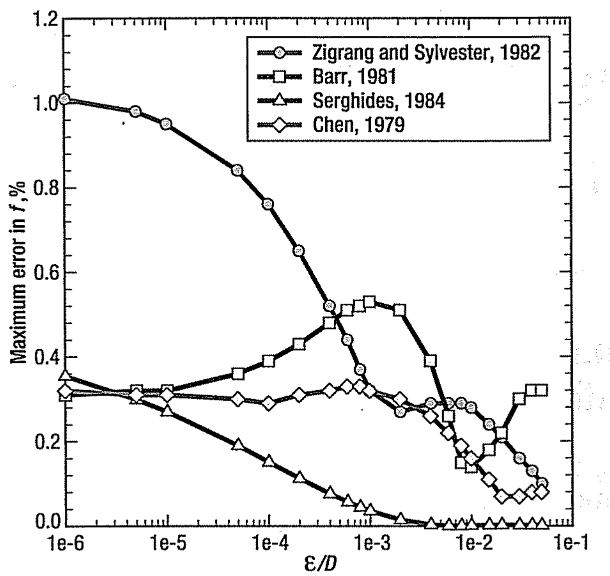

FIG. 1 Error associated with computing friction factor from correlations 1–4 in Table 1. A total of 500 *f* values in the 4,000 < Re < 10^8 range were computed for each ε/D and only the maximum errors are shown.

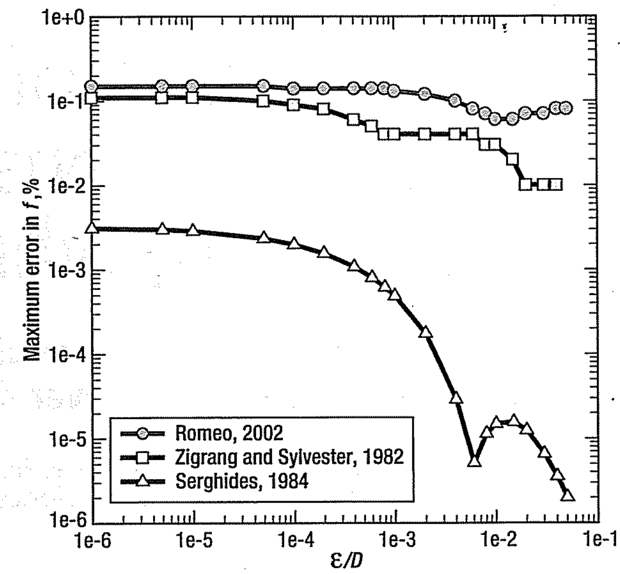

FIG. 2 Error associated with computing friction factor from correlations 5–7 in Table 1. A total of 500 *f* values in the 4,000 < Re < 10^8 range were computed for each ε/D and only the maximum errors are shown.

Thus, two versions of Eq. 2 are possible depending upon the choice of δ:

$$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta_{LA}\right] \quad (4)$$

$$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta_{CFA}\right] \quad (5)$$

A comparison of the error properties of various iterative empirical approximations of Eq. 1 is presented along with error in f estimates from Eqs. 4 and 5.

Comparison with empirical approximations. The accuracy of Eqs. 4 and 5 and the empirical iterative approximations of Eq. 1 were determined over a rectangular space of ε/D and *Re* values. A set of 20 ε/D values corresponding to those used by Moody[1] were selected that spanned a range from 10^{-6} to 5×10^{-2}. For each ε/D value 500 values of *Re*, distributed uniformly in the logarithmic space over $4{,}000 < Re < 10^8$, were chosen. Accuracy of f values at these 10,000 points (20 × 500 grid of ε/D and *Re* values) was determined by comparing them with those obtained from the highly accurate mathematically equivalent form.[11]

A total of 10,000 f values and their associated error were determined over the 20 × 500 grid of ε/D and *Re* values, and the maximum error values are shown in Table 1. While not all Table 1 correlations are valid over the entire *Re* range (4,000 < Re < 10^8), comparison was intentionally made over this extended range to reflect operational conditions. The maximum f error ranged from 1.01 to 3.10×10^{-3}% with the Serghides correlation[5] being the most accurate. Correlations 8 and 9, which are derived from an explicit mathematically equivalent representation of Eq. 1, were characterized by maximum f errors of 3.64×10^{-4} and 1.04×10^{-10}%, both better than the best available iterative approximation.

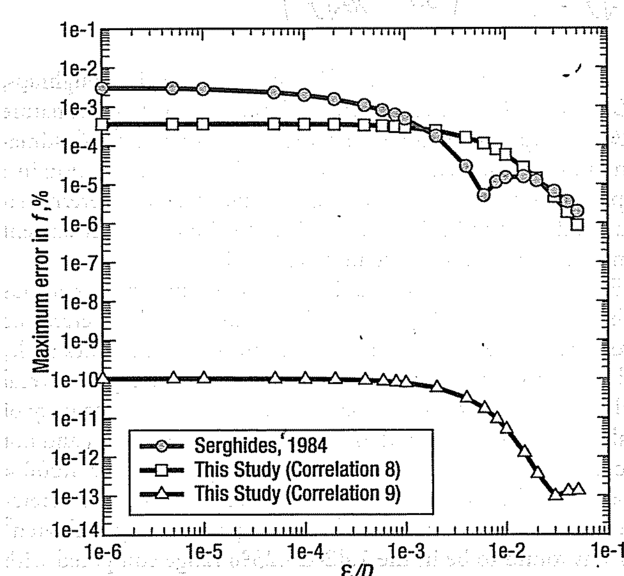

FIG. 3 Error associated with computing friction factor from correlations 7–9 in Table 1. A total of 500 *f* values in the 4,000 < Re < 10^8 range were computed for each ε/D and only the maximum errors are shown.

Accuracy of the correlations in Table 1 is shown in Figs. 1 and 2 where the maximum percentage f error is shown at varying ε/D values. For each ε/D value, 500 f values were determined at

TABLE 1. Comparison of errors in f estimates from various iterative approximations of the Colebrook-White equation

	Correlation	Maximum absolute error in f, %	Reference
1	$\frac{1}{\sqrt{f}} = -2\log\left\{\frac{\varepsilon/D}{3.7} - \frac{5.02}{Re}\log\left(\frac{\varepsilon/D}{3.7} + \frac{13}{Re}\right)\right\}$	1.01	6
2	$\frac{1}{\sqrt{f}} = -2\log\left\{\frac{\varepsilon/D}{3.7} + \frac{4.518\log\left(\frac{Re}{7}\right)}{Re\left(1+\frac{1}{29}Re^{0.52}\left(\frac{\varepsilon}{D}\right)^{0.7}\right)}\right\}$	0.53	8
3	$f = \left\{4.781 - \frac{(A-4.781)^2}{B-2A+4.781}\right\}^{-2}$ $A = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{12}{Re}\right)$ $B = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51A}{Re}\right)$	0.36	5
4	$\frac{1}{\sqrt{f}} = -2\log\left\{\frac{\varepsilon/D}{3.7065} - \frac{5.0452}{\text{Re}} \times \log\left(\frac{(\varepsilon/D)^{1.1098}}{2.8257} + \frac{5.5806}{Re^{0.8981}}\right)\right\}$	0.33	9
5	$\frac{1}{\sqrt{f}} = -2\log\left\{\frac{\varepsilon/D}{3.7065} - \frac{5.0272}{Re}\log\left(\frac{\varepsilon/D}{3.827} - \frac{4.567}{Re}\log\left(\left(\frac{\varepsilon/D}{7.7918}\right)^{0.9924} + \left(\frac{5.3326}{208.815+Re}\right)^{0.9345}\right)\right)\right\}$	0.15	4
6	$\frac{1}{\sqrt{f}} = -2\log\left\{\frac{\varepsilon/D}{3.7} - \frac{5.02}{Re}\log\left(\frac{\varepsilon/D}{3.7} - \frac{5.02}{Re}\log\left(\frac{\varepsilon/D}{3.7} + \frac{13}{Re}\right)\right)\right\}$	0.11	6
7	$f = \left(A - \frac{(B-A)^2}{C-2B+A}\right)^{-2}$ $A = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{12}{Re}\right)$ $B = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51A}{Re}\right)$ $C = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51B}{Re}\right)$	3.10×10^{-3}	5
8	$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta_{LA}\right]$	3.64×10^{-4}	This study
9	$\frac{1}{\sqrt{f}} = a\left[\ln\left(\frac{d}{q}\right) + \delta_{CFA}\right]$	1.04×10^{-10}	This study

500 logarithmically spaced Re values in the $4{,}000 < Re < 10^8$ range and the maximum values are shown in Figs. 1 and 2. The Serghides equation (correlation 7) with a maximum error of 3.1×10^{-3}% is the best available empirical approximation. Fig. 3 shows a comparison of f error profiles for the Serghides equation with those from Eqs. 4 and 5. Maximum f error from Eqs. 4 and 5 were 3.64×10^{-4} and 1.04×10^{-10}%, respectively, and this improved accuracy is reflected in Fig. 3. **HP**

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