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THE TRAP OF THE ANTICIPATED RESULT
OR
THE TIME REQUIRED TO EMPTY A RESERVOIR

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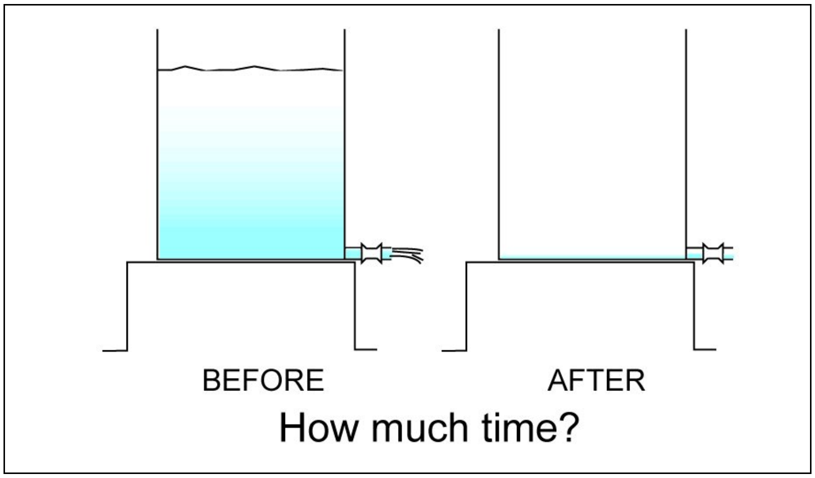

Figure 1 How much time does it take to empty a reservoir.

Years ago, approximately 1994, I published a solution to this problem on my web site and it was wrong. This was pointed out to me by a gentleman whose name I have forgotten. He said that he didn't remember much about derivatives but that the graph of the solution that I presented had to be wrong.

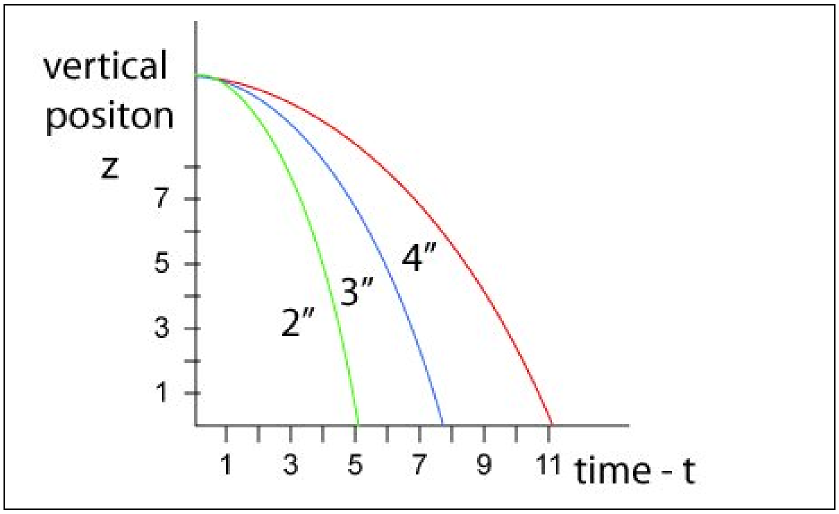

Figure 2 Graph of wrong solution to how much time is required to empty a tank for different outlet sizes.

He stated quite rightly that the level would not drop slowly at first and then increase it's rate of descent but that it would more likely be moving fast at the beginning and slow down as the tank emptied. I agreed with him, but I could not find the flaw in my reasoning.

After a few half hearted attempts I quit trying to find the error and moved on. I left the wrong solution on the website for a number of years, my apologies to those who were lead astray. It was eventually removed from the site.

The old article can be found at this link: http:www.lightmypump.com/SOLVED PROBLEMS.pdf.

Recently I had an occasion to examine this problem once again in relation to my job. I went to my previous writings and once again it seemed to make perfect sense to me. I took the equation that I had developed and plugged some values into an Excel spreadsheet and plotted a graph. I obtained the same graph as before, the one in Figure 2. At this point, something was nagging me and I couldn't remember what but it eventually came back to me, the gentleman's warning that something was wrong. OK, this time I was going to do something about it. I felt that probably the theory was right but that perhaps there was something wrong with the initial conditions. Initial conditions are the conditions of the object whose movement you are trying to explain, they are often about position and velocity such as at time t = 0 the position and velocity are such a and such. A good example of initial conditions are those associated with the free falling body problem. When you throw an object upwards it will initially rise and then fall to the ground, the shape of the path is a parabola. The initial conditions required are the angle of the initial trajectory and the initial velocity at time t=0.

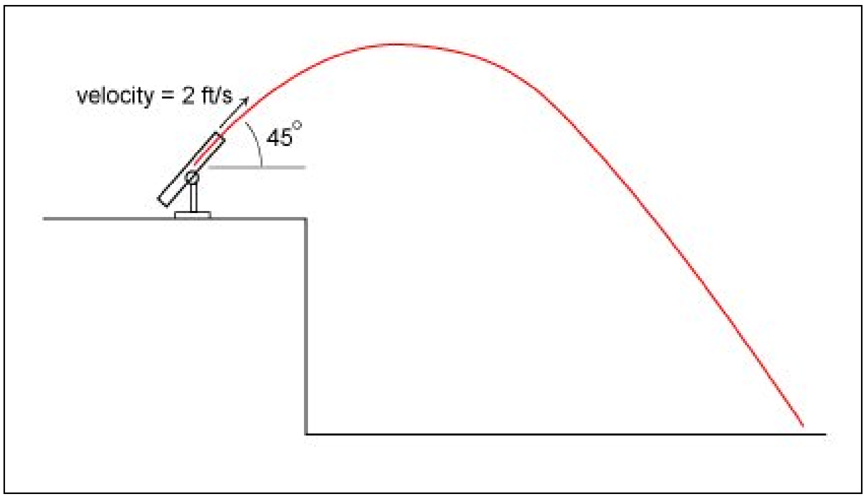

Figure 3 Falling body problem with initial conditions.

There is a problem that occurs often to people who do these types of calculations which is that the result obtained is plausible. For example in the free falling body problem if your results, with the proper initial conditions, predict that the object would travel 100 feet in the horizontal direction and you expected it to be around 100 feet then you would feel comfortable with the results. Furthermore if you varied the angle and the initial velocity and got further plausible results that would clinch it. Unfortunately mistakes often provide plausible results and that's what happened to me. We need several ways to look at the results to avoid this trap.

Back to my problem, I knew my solution was wrong because the graph that was produced from the results was likely wrong (see Figure 2). So I decided to sketch what I thought the shape of the graph should be and see what initial conditions would be required. I also did the graph of the velocity because that would be an important consideration.

My first assumption about the initial conditions was that at time t=0, the velocity would be zero. This leads to the 2 graphs you see below.

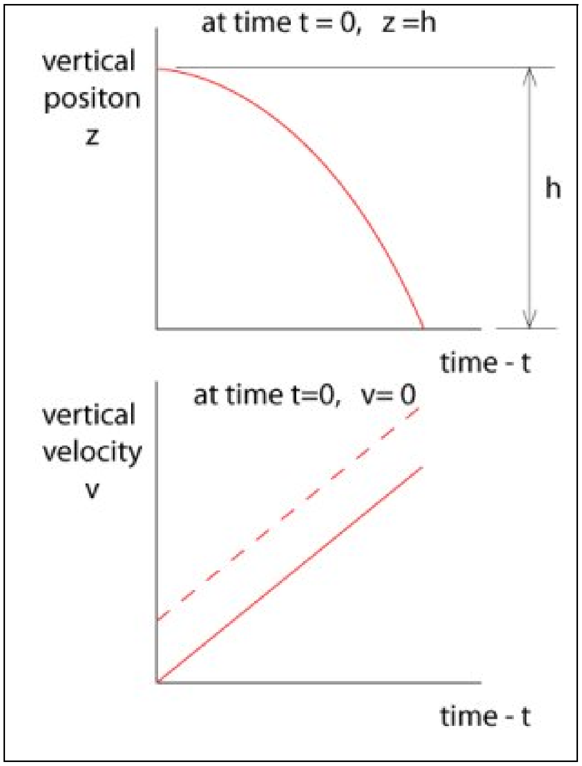

Figure 3 Initial conditions on position and velocity of the water level, attempt no. 1.

The top graph seems to make sense since the position of the surface gradually falls to zero. The velocity graph is constructed by taking the slope of all the points on the position graph, also known as taking the derivative of its equation. The slope of the curve in the top graph at the time $t = 0$ is zero, this means that the velocity can be either zero or a certain fixed number represented by the position of the dashed line as it intersects the z axis. As the slope increases in the top graph to its maximum when $z = 0$ so does the velocity. As a first approximation we can represent the behavior of velocity as a straight line which may or may not start at zero, this will depend on the initial conditions that we stipulate.

Of course, this is where my mistake lies because I do not believe in the top graph and I need to take a different approach to get out of this loop.

I decide to graph my expectations for velocity and see what the impact is on the position graph since they are both directly linked.

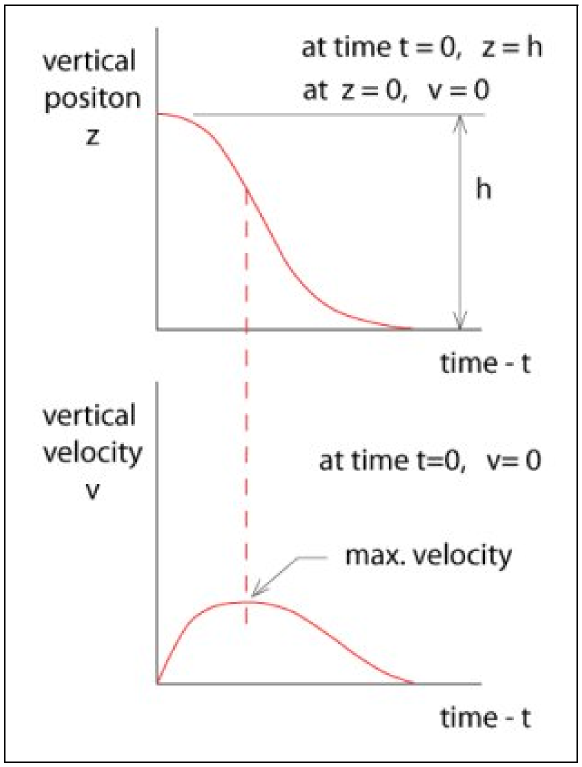

Figure 4 Initial conditions on position and velocity of the water level, attempt no.2.

It seemed to me that the velocity would start at zero and then eventually end at zero when the tank was empty. In between it would have to increase to some sort of maximum and then fall towards zero and this is what the bottom graph of Figure 4 shows. This behavior requires two initial conditions: one at time $t = 0$ when $z = h$, and second at position $z = 0$ where $v = 0$ and then in the middle it is required that the velocity be maximum at some unknown time and position. This is a 3rd condition and the problem only requires two conditions to be solved so that this cannot fit the model that is proposed. The model will be discussed shortly.

Then it dawned on me, the model that I am using is known as the steady-state model, in other words it assumes that fluid is flowing continuously and I am only capturing it at a certain moment of time that I call zero where z = h, there is no ability in this model to have the fluid start from rest, accelerate to a high rate and then decrease towards zero. I need to take this into account in the position and velocity graphs. My initial conditions have to fit the limitations of the model and given this the only graphs that make sense are those shown in Figure 5 below.

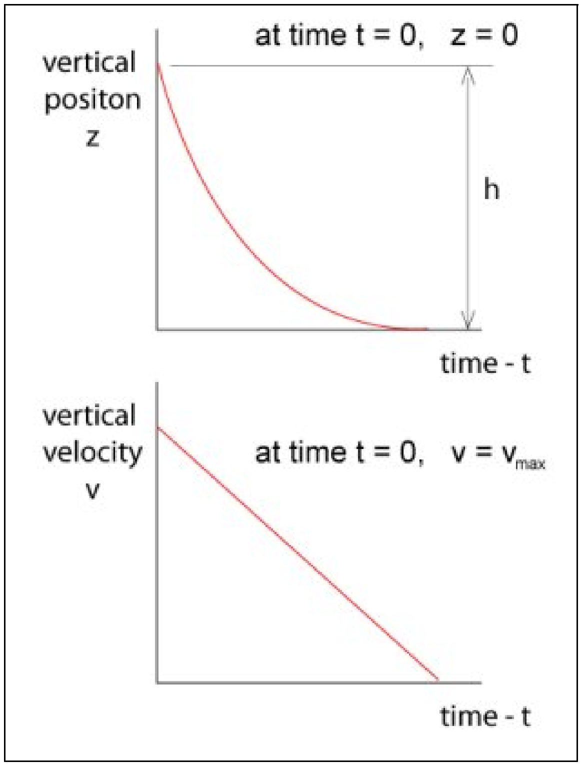

Figure 5 Initial conditions on position and velocity of the water level, final solution no.3.

The initial conditions are: at time t = 0 z = 0, and at time t = 0 v = v_{max}. v_{max} is the maximum flow rate which occurs at the beginning and depends on the height h of the liquid level and the size of the outlet. The theory can easily calculate that.

So this is a great example on how to look at a problem in different ways and avoid the trap of anticipated results.

The model, the equations and the results

The model used is a simple one that assumes the fluid has a low viscosity like water or similar fluid where the friction characteristics are well known especially resistance factors associated with valves and fittings.

The easiest way to formulate this problem is to do a balance of energy, the energy available or coming into the system has to be equal to the energy lost or going out of the system. This can be expressed by this formula:

$$z_1 = H_F + \frac{1}{2g}\left(v_2^{\ 2} - v_1^{\ 2}\right) \qquad (1)$$

where z_1 is the static head and H_F is the outlet friction loss, v_2 the velocity at the outlet and v_1 the velocity of the liquid surface. z_1 will vary with time as the tank drains and an equation for this variation is developed.

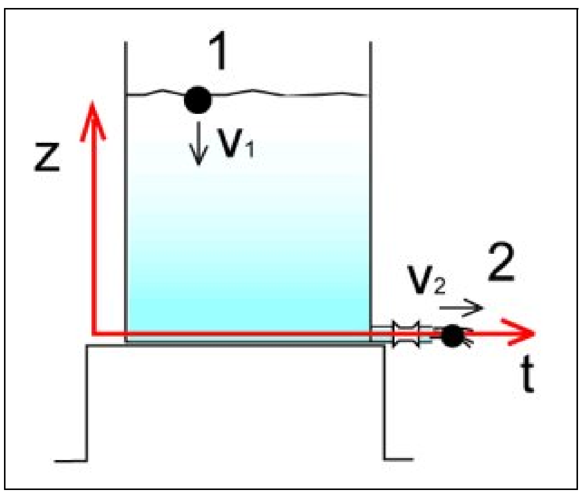

Figure 6 Position of the different variables to be examined in the model.

You will have noticed that the above equation states that z must be equal to H_F, the friction loss, at all times, how is this an energy balance? What are the units of these terms? z will be in feet, how can this be energy? Well actually it is not exactly energy but something very similar, z is energy per weight of fluid displaced, the units of energy per unit weight simplifies to feet and we call this head, in this particular case it is static head. We can show this in more detail by looking at the units, energy can be expressed in lbf –ft or pound feet and weight are lbfs which are pounds. Therefore:

$$\frac{energy}{weight} = \frac{lbf - ft}{lbf} = ft$$

Therefore z is in fact an energy term and these terms are known as head in the fluid world, and this is where terms like static head and total head come from. And you can see how much simpler this makes things since instead of using two terms such as energy and weight we can use an equivalent term with a simple unit of length which we call head.

How does the term H_F become a head term? In our problem the friction will primarily be at the outlet where the liquid has to pass through a narrow opening. The friction of the liquid against the walls of the tank will be very small because the velocity is low at that point.

The first source of friction that we have is related to movement of the liquid into a narrow opening, this is known as exit loss and is expressed as:

$$H_e = K_e \times \frac{v^2}{2g}$$

K_e known as a K factor is a constant associated with a particular type of restriction, in this case an exit restriction and its value is 1. v is the velocity of the fluid particle in the restriction; the units are *ft/s* and g the acceleration due to gravity which is 32.17 ft/s^2 and K_e has no units. How is H_e a head term? For H_e to be a head term it has to be in feet:

$$H_e(ft) = K_e \times \frac{v^2(ft/s)^2}{2 \times 32.17(ft/s^2)}$$

$v^2/2g$ is what's known as velocity head and this is akin to kinetic energy or the energy of movement. This is the energy of a ball that is thrown at someone, the catcher has to absorb the ball's kinetic energy.

The traditional way to write kinetic or velocity energy is KE = ½ m v^2, where m is the mass of the object and v its velocity. How can velocity energy be turned into velocity head for fluid particles. Remember that head is energy per unit weight of fluid.

$$velocity\ head = \frac{KE}{W} = \frac{KE}{mg} = \frac{1}{2} \times \frac{mv^2}{mg} = \frac{v^2}{2g}$$

where W is the weigh of the object which is equal to mg and ½ mv^2 over mg is equal $v^2/2g$.

So the velocity head is the energy of the fluid particles at for example point 2 and K_e is a multiplier or constant that gives the amount of energy that is lost at the exit.

$$H_e = K_e \times \frac{v_2^{\;2}}{2g}$$

H_F is made up of two types of friction: exit friction and valve friction:

$$H_F = H_e + H_v$$

where H_e is the exit friction head which we have already discussed and H_v the valve friction head. The valve friction head factor will depend on the type of valve, in our case we will be using a valve that opens fully such as a knife gate valve and does not offer any friction in that position therefore $H_v = 0$.

The exit friction is expressed as a portion of the exit velocity head multiplied by a K factor. The K factor which we call K_e for exit is equal to 1. So that H_e is:

$$H_e = \frac{1}{2g} K_e v_2^{\;2}$$

Therefore equation (1) becomes:

$$z_1 = \frac{1}{2g} K_e v_2^{\;2} + \frac{1}{2g}\left(v_2^{\;2} - v_1^{\;2}\right) \qquad (2)$$

The term 1/2g ($v_2^2 - v_1^2$) is the difference in velocity energy between what's coming into the system vs. what is coming out.

Since we want a relationship between z_1 (i.e. the position of the liquid surface) and time t we need to find out the relationship between v_2 and v_1.

We know that the same flow rate (i.e. gals/min) will pass through the tank as will pass through the outlet. Remember that liquids are incompressible and that if one gallon comes out the exit a corresponding volume will have come out the tank and this allows us to get a relationship between v_1 and v_2. v_1 and v_2 will vary because the tank is emptying but at any given instant in time an equal volume will come out of the exit that comes out of the tank. The flow rate Q will depend on the cross-sectional area A and the velocity v:

$$Q = A \times v$$

In particular A_1 is the cross-sectional area of the tank and A_2 the cross-sectional area of the pipe or exit.

Therefore:

$$Q = A_1 \times v_1 = A_2 \times v_2$$

which in turn means that:

$$v_2 = \frac{A_1}{A_2} \times v_1$$

We can replace v_2 in equation (2):

$$z_1 = \frac{1}{2g}\left[(K_e + 1)\left(\frac{A_1}{A_2}\right)^2 - 1\right] v_1^{\,2} \qquad (3)$$

The term $1/2g \times ((K_e+1) \times (A_1/A_2)^2 - 1))$ is a constant which we can replace with one term G to make our life easier.

$$G = \frac{1}{2g}\left[(K_e + 1)\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

Therefore:

$$z_1 = G \times v_1^{\,2} \qquad (4)$$

Here is where the calculus comes in if we want to make prediction about how the system will behave from start to finish, we need to come up with a way of handling quantities that vary continuously such as the surface position z_1 which varies with time. If we take the differential of z_1 we will get velocity because that's what the differential does, it provides the rate of change of a quantity which in this case is its velocity so that $dz_1/dt = v_1$.

The differential of v_1^2 is $2 \times v_1 \times dv_1/dt$.

Therefore the differential of equation (4) is:

$$\frac{dz_1}{dt} = G \times 2 \times v_1 \times \frac{dv_1}{dt}$$

By definition dz_1/dt is equal to v_1. Therefore:

$$v_1 = G \times 2 \times v_1 \times \frac{dv_1}{dt}$$

or

$$1 = G \times 2 \times \frac{dv_1}{dt}$$

dv_1/dt by definition is d_2z_1/dt^2, therefore the above equation becomes:

$$1 = G \times 2 \times \frac{d_2 z_1}{dt^2}$$

or

$$\frac{d_2 z_1}{dt^2} = \frac{1}{2G} \tag{5}$$

This is a simple second order differential equation that can be integrated twice to provide the solution for z_1 as a function of time.

The first integration gives:

$$\frac{dz_1}{dt} = \int \frac{1}{2G} dt + A = \frac{1}{2G} t + A$$

Each time we integrate we introduce a constant and this will correspond to one of our initial conditions. After the second integration we get the result we are looking for:

$$z_1 = \int \left(\frac{1}{2G} t + A \right) dt + B = \frac{1}{2G} \frac{t^2}{2} + At + B$$

or

$$z_1 = \frac{t^2}{4G} + At + B \tag{6}$$

where A and B are constants that are determined by the initial conditions. These conditions are at t = 0, z_1 = h and also at t = 0, v= v_{max}.

The first initial condition gives us B = h, equation (6) becomes:

$$z_1 = \frac{t^2}{4G} + At + h$$

From the first integration we know that the velocity is:

$$v_1 = \frac{dz_1}{dt} = \frac{1}{2G}t + A$$

The second initial condition gives us A =- v_{max}. Why – v_{max} and not v_{max}. Because if you look at Figure 5 you will see that the slope for velocity is negative since the velocity is decreasing.

The complete equation is then:

$$z_1 = \frac{t^2}{4G} - v_{\max} t + h \qquad (7)$$

with $G = \frac{1}{2g} K \times \left(\frac{A_1}{A_2}\right)^2$

All we need to do now is determine v_{max} and we have a complete solution. From equation (4), the value of v_{max} is:

$$v_{\max} = \left(\frac{h}{G}\right)^{1/2}$$

We are now ready to build a spreadsheet for equation (7), by providing some values for t and doing the calculation for z based on equation (7) and then plot a graph of the result.

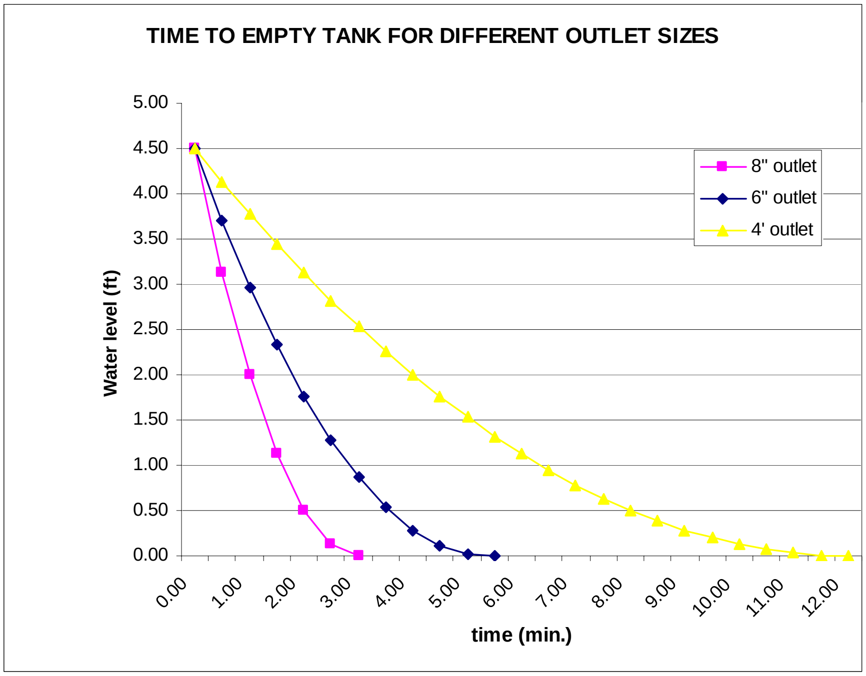

Figure 7 Results of a calculation using equation (7).